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# COHERENCE FOR BICATEGORIES AND INDEXED CATEGORIES *

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## 1. Introduction

Indexed categories are useful in dealing with families, more precisely families indexed by objects in some suitable category. The fundamental operation is the substitution of a morphism $\alpha$ in that category into a family, giving a functor $\alpha^*$. However, in the important cases, $\beta^*\alpha^*$ is not equal to $(\alpha\beta)^*$ but only isomorphic to it, and for everything to work properly these isomorphisms must satisfy coherence conditions. Then, when working with indexed categories, these isomorphisms keep coming up and large diagrams involving them must commute. These diagrams are such that it would be very surprising if they didn't, and in the first stages of work it is always assumed that they do. But in the end, their commutativity must be checked, and if indexed categories are to be useful as a tool for studying topoi and related categories, an efficient way must be found to deal with these diagrams. That is to say we want a coherence theorem.

The first categorical coherence theorem was MacLane's for monoidal categories [9] (see also [10]). A monoidal category is essentially the same as a bicategory with one object, and this coherence theorem extends without trouble to bicategories. This, together with the relevant definitions, is given in Section 2.

In Section 3, the definitions of indexed category, functor, and natural transformation, and the coherence conditions they must satisfy are set down carefully. Then a bicategory is constructed for which the previous theorem specializes to a coherence theorem for indexed categories.

However the canonical maps thus obtained are not sufficient to efficiently carry

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out the calculations we want in indexed categories, so in Section 4 we extend our class of canonical maps. We then get a coherence theorem for these.

In the last section we give some examples of the coherence theorem in action.

## 2. Bicategories

A *bicategory* $\mathfrak{B}$ (Bénabou [1]) consists of objects $A, B, C, \ldots$ and for each pair of objects, $A$ and $B$, a category $\mathfrak{B}(A, B)$. The objects of $\mathfrak{B}(A, B)$ are called morphisms (or 1-cells) of $\mathfrak{B}$ and the morphisms of $\mathfrak{B}(A, B)$ are the 2-cells of $\mathfrak{B}$. Composition in $\mathfrak{B}(A, B)$ is denoted by juxtaposition. For any three objects, $A, B, C$, there is given a composition functor

$$\mathfrak{B}(B,C)\times\mathfrak{B}(A,B)\to\mathfrak{B}(A,C)$$
$$(G,F)\mapsto G\otimes F$$

which is unitary and associative up to coherent isomorphisms.

The functoriality of $\otimes$ means that the $\otimes$ of two identity 2-cells is an identity and that we have the middle four interchange law, i.e.

$$1_G\otimes 1_F=1_{G\otimes F} \quad \text{and} \quad (u_2u_1)\otimes(t_2t_1)=(u_2\otimes t_2)(u_1\otimes t_1)$$

for suitably composable morphisms $u_i$ and $t_i$. That $\otimes$ is associative means that for any 1-cells

$$A\xrightarrow{F}B\xrightarrow{G}C\xrightarrow{H}D$$

we are given an isomorphism

$$a_{H,G,F}:(H\otimes G)\otimes F\to H\otimes(G\otimes F)$$

natural in $F$, $G$, and $H$. To say that $\otimes$ is unitary means that for every object $A$ we are given a morphism $1_A: A\to A$ and for every morphism $F: A\to B$ isomorphisms

$$b_F: F\otimes 1_A\to F, \qquad c_F: 1_B\otimes F\to F$$

natural in $F$. These isomorphisms are required to satisfy the following coherence conditions:

(B1)
$$\begin{array}{ccc}
((K\otimes H)\otimes G)\otimes F & \xrightarrow{a_{K\otimes H,G,F}} & (K\otimes H)\otimes(G\otimes F) \\
\downarrow{\scriptstyle a_{K,H,G}\otimes F} & & \\
(K\otimes(H\otimes G))\otimes F & & \downarrow{\scriptstyle a_{K,H,G\otimes F}} \\
\downarrow{\scriptstyle a_{K,H\otimes G,F}} & & \\
K\otimes((H\otimes G)\otimes F) & \xrightarrow[K\otimes a_{H,G,F}]{} & K\otimes(H\otimes(G\otimes F))
\end{array}$$

(B2)
$$\begin{array}{ccc} (G\otimes 1_B)\otimes F & \xrightarrow{a_{G,1_B,F}} & G\otimes(1_B\otimes F) \\ & {\scriptstyle b_G\otimes F}\searrow \quad \swarrow{\scriptstyle G\otimes c_F} & \\ & G\otimes F\,. & \end{array}$$

A bicategory with a single object is essentially a monoidal category, so we view bicategories as generalized monoidal categories. In fact the coherence conditions given here are Kelly's refinement of those for monoidal categories [7, p. 21] and are somewhat different from [1].

A bicategory in which all $a, b, c$, are identities is called a *2-category*.

We want to prove a coherence theorem for bicategories, which would say that any diagram made up of $a, b$, and $c$'s commutes. In a given bicategory it may happen that certain relations, such as $G\otimes F=G'\otimes F'$, hold by accident and this must be avoided in order to obtain our theorem. So, following Laplaza [8], we construct a new bicategory $\mathfrak{W}\mathfrak{B}$ as follows. The objects of $\mathfrak{W}\mathfrak{B}$ are the same as those of $\mathfrak{B}$. The morphisms of $\mathfrak{W}\mathfrak{B}$ are non-associative words of composable morphisms of $\mathfrak{B}$. These are defined inductively by:

(1) If $F: A\to B$ is a morphism of $\mathfrak{B}$, then $\ulcorner F\urcorner : A\to B$ is a morphism of $\mathfrak{W}\mathfrak{B}$.

(2) If $W_1: A\to B$ and $W_2: B\to C$ are morphisms of $\mathfrak{W}\mathfrak{B}$, then so is $(W_2\otimes W_1): A\to C$.

Now, define a function $\varepsilon$ which associates to each morphism of $\mathfrak{W}\mathfrak{B}$ the morphism of $\mathfrak{B}$ obtained by removing the $\ulcorner\ \urcorner$ and evaluating. Thus

$$\varepsilon(\ulcorner F\urcorner)=F \quad\text{and}\quad \varepsilon(W_2\otimes W_1)=\varepsilon(W_2)\otimes\varepsilon(W_1).$$

A 2-cell $f: W\to W'$ in $\mathfrak{W}\mathfrak{B}$ is defined to be a 2-cell $f:\varepsilon W\to\varepsilon W'$ in $\mathfrak{B}$. Composition of 2-cells is performed in $\mathfrak{B}$ and thus $\varepsilon$ becomes an equivalence of categories $\mathfrak{W}\mathfrak{B}(A,B)\to\mathfrak{B}(A,B)$. There is an obvious $\otimes$ on the morphisms of $\mathfrak{W}\mathfrak{B}$ which trivially extends to the 2-cells. If $a_{W_3,W_2,W_1}, b_W, c_W$ are taken to be $a_{\varepsilon W_3,\varepsilon W_2,\varepsilon W_1}, b_{\varepsilon W}, c_{\varepsilon W}$ respectively, $\mathfrak{W}\mathfrak{B}$ becomes a bicategory and $\varepsilon$ gives a strict homomorphism of bicategories $\mathfrak{W}\mathfrak{B}\to\mathfrak{B}$.

Let $\mathfrak{C}$ be the smallest subbicategory of $\mathfrak{W}\mathfrak{B}$ with the same objects and the same morphisms. Thus $\mathfrak{C}$ must contain all instances of $a, b, c$, their inverses and must be closed under composition and $\otimes$. The 2-cells of $\mathfrak{C}$ are called the *canonical 2-cells* of $\mathfrak{W}\mathfrak{B}$ (or of $\mathfrak{B}$). They are all invertible.

**Theorem.** *To each morphism $W: A\to B$ in $\mathfrak{W}\mathfrak{B}$ there is associated a morphism $\sigma W: A\to B$ and a canonical 2-cell $s_W: W\to\sigma W$. For $W, W': A\to B$ there is at most one canonical 2-cell $W\to W'$ and there is one if and only if $\sigma W=\sigma W'$.*

**Proof** (outline). Although the formal set up is different, the substantial part of the proof is the same as for the monoidal case [10, p. 162] so we content ourselves with giving the general idea.

We call $\sigma W$ the *standard form* of $W$, and it is obtained by dropping all identities (unless $W$ is of the form $\ulcorner 1_A \urcorner$) and moving all parentheses so that they end at the right. Thus

$$\sigma((K\otimes(H\otimes G))\otimes(1_B\otimes F))=K\otimes(H\otimes(G\otimes F)).$$

Then $s_W: W\to\sigma W$ is a composite of *elementary canonical maps* (those containing only one instance of $a$, $b$, or $c$, and no inverses); one involving $b$ or $c$ for each identity dropped and a number involving $a$ required to move the parentheses to the right as in the proof of the 'general associative law'. If $\sigma W=\sigma W'$, then $s_{W'}^{-1}s_W: W\to W'$ is a canonical map. Conversely, if there is a canonical map $W\to W'$, then $\sigma W=\sigma W'$ as can be seen by inspecting the domains and codomains of the elementary canonical maps.

For uniqueness we show that for any canonical map $c: W\to W'$

$$\begin{array}{ccc} W & \xrightarrow{\ c\ } & W' \\ {\scriptstyle s_W}\downarrow & & \downarrow{\scriptstyle s_{W'}} \\ \sigma W & = & \sigma W' \end{array}$$

commutes. It is sufficient to show this for $c$ elementary, as every canonical map is a composite of elementary ones and their inverses.

Then the proof is by simultaneous induction on the *length* and *rank* of $W$. These quantities are defined recursively by

$$\mathrm{length}(\ulcorner F\urcorner)=1,$$

$$\mathrm{length}(W_1\otimes W_2)=\mathrm{length}(W_1)+\mathrm{length}(W_2),$$

and

$$\mathrm{rank}(\ulcorner 1_A\urcorner)=1,$$

$$\mathrm{rank}(\ulcorner F\urcorner)=0 \quad \text{for } F\neq 1,$$

$$\mathrm{rank}(W_1\otimes W_2)=\mathrm{rank}(W_1)+\mathrm{rank}(W_2)+\mathrm{length}(W_1)-1.$$

The proof is then wrapped up by considering cases just as in *loc. cit.* □

This proof can also be viewed as an application of the Newman Diamond Lemma (see [12] or [4]). The theorem itself is a special case of a much more general coherence theorem formulated at the end of Bénabou's thesis [2].

## 3. Indexed categories

Let **S** be a category with finite limits. For example **S** might be an elementary topos, which is where many of the applications lie. **S**-indexed categories, functors, and natural transformations were defined in [13, p. 7,8] but the concepts were somewhat obscured by the introduction of functors 'defined up to' certain specified isomorphisms. The definitions given below are somewhat different in that these are avoided. In the terminology of Gray [5, p. 40–45], an **S**-indexed category is nothing but a homomorphic pseudo-functor $\mathbf{S}^{op} \to \mathfrak{Cat}$, an **S**-indexed functor a quasi-natural transformation in which the $\sigma$ are isomorphisms, and an indexed natural transformation a modification. Pseudo-functors were introduced by Grothendieck. See [6, p. 145–194], where their relationship with fibrations is also discussed. They are also a special case of morphisms of bicategories [1, p. 47].

Because it is difficult to find all the definitions we need, and in the generality we want, in one place in the literature we give the definitions in full. This will also serve to fix notation.

An **S**-*indexed category* $\mathscr{A}$ consists of the following data:

(1) for each $I \in \mathbf{S}$, a category $\mathbf{A}^I$,

(2) for each $\alpha : J \to I \in \mathbf{S}$ a functor $\alpha^* : \mathbf{A}^I \to \mathbf{A}^J$,

(3) for each composable pair $K \xrightarrow{\beta} J \xrightarrow{\alpha} I \in \mathbf{S}$, a natural isomorphism $\phi_{\alpha,\beta} : \beta^*\alpha^* \to (\alpha\beta)^*$,

(4) for each $I \in \mathbf{S}$, a natural isomorphism $\psi_I : (1_I)^* \to 1_{\mathbf{A}^I}$, subject to the following coherence conditions:

(C1) for each composable triple $L \xrightarrow{\gamma} K \xrightarrow{\beta} J \xrightarrow{\alpha} I$ in **S**

$$\begin{array}{ccc} \gamma^*\beta^*\alpha^* & \xrightarrow{\gamma^*\phi_{\alpha,\beta}} & \gamma^*(\alpha\beta)^* \\ {\scriptstyle \phi_{\beta,\gamma}\alpha^*}\downarrow & & \downarrow{\scriptstyle \phi_{\alpha\beta,\gamma}} \\ (\beta\gamma)^*\alpha^* & \xrightarrow[\phi_{\alpha,\beta\gamma}]{} & (\alpha\beta\gamma)^* , \end{array}$$

and

(C2) for each $\alpha : J \to I \in \mathbf{S}$,

$$\phi_{1_I,\alpha} = \alpha^*\psi_I : \alpha^* 1_I^* \to \alpha^*.$$

When condition (C1) is rewritten as $\phi_{\alpha\beta,\gamma} \circ (\gamma^*\phi_{\alpha\beta}) = \phi_{\alpha,\beta\gamma} \circ (\phi_{\beta,\gamma}\alpha^*)$ it is the analogue of the 2-cocycle condition in the cohomology of groups, with two sided operators. This analogy has also been noted by B. Mitchell [11].

If $\mathscr{A}$ and $\mathscr{B}$ are indexed categories, an *indexed functor* $F : \mathscr{A} \to \mathscr{B}$ consists of the following data:

(1) for each $I \in \mathbf{S}$, a functor $F^I : \mathbf{A}^I \to \mathbf{B}^I$,

(2) for each $\alpha : J \to I \in \mathbf{S}$ a natural isomorphism $\theta_\alpha : \alpha^* F^I \to F^J \alpha^*$, subject to the condition

(F) for each composable pair $K \xrightarrow{\beta} J \xrightarrow{\alpha} I$ in $\mathbf{S}$,

$$\begin{array}{ccc}
\beta^*\alpha^*F^I & \xrightarrow{\phi_{\alpha,\beta}F^I} & (\alpha\beta)^*F^I \\
{\scriptstyle \beta^*\theta_\alpha}\downarrow & & \\
\beta^*F^J\alpha^* & & \downarrow{\scriptstyle \theta_{\alpha\beta}} \\
{\scriptstyle \theta_\beta\alpha^*}\downarrow & & \\
F^K\beta^*\alpha^* & \xrightarrow[F^K\phi_{\alpha,\beta}]{} & F^K(\alpha\beta)^* .
\end{array}$$

If $F, G : \mathscr{A} \to \mathscr{B}$ are two indexed functors, an *indexed natural transformation* $t : F \to G$ consists of a natural transformation $t^I : F^I \to G^I$ for each $I \in \mathbf{S}$, such that

(NT) for each $\alpha : J \to I$,

$$\begin{array}{ccc}
\alpha^*F^I & \xrightarrow{\alpha^*t^I} & \alpha^*G^I \\
{\scriptstyle \theta_\alpha}\downarrow & & \downarrow{\scriptstyle \theta_\alpha} \\
F^J\alpha^* & \xrightarrow[t^J\alpha^*]{} & G^J\alpha^* .
\end{array}$$

**Note.** In the above-mentioned references there is an extra condition on pseudo-functors which would be, in our case,

(C3) for every $\alpha : J \to I$,

$$\phi_{\alpha, 1_J} = \psi_J\alpha^* : 1_J^*\alpha^* \to \alpha^*$$

and an extra one on quasi-natural transformations which would be

(F2) for every $I$,

$$\begin{array}{ccc}
1_I^*F^I & \xrightarrow{\theta_{1_I}} & F^I1_I^* \\
& {\scriptstyle \psi_IF^I}\searrow \quad \swarrow{\scriptstyle F^I\psi_I} & \\
& F^I . &
\end{array}$$

Here, where the $\phi$ and $\psi$ are isomorphisms, C3 follows from C1 and C2 and F2 follows from F and C2, as we shall see after we have constructed the bicategory associated with S-indexed categories.

As mentioned in [5, p. 45], S-indexed categories, functors, and natural transformations form a 2-category $\mathbf{S}$-$\mathfrak{ind}$.

The vertical structure of $\mathbf{S}$-$\mathfrak{ind}$ is defined in terms of that of $\mathfrak{Cat}$, i.e. if $t : F \to F'$ and $t' : F' \to F''$ are indexed natural transformations, $t't : F \to F''$ is defined by

$(t't)^I = t'^I t^I$, and $1_F$ by $(1_F)^I = 1_{F^I}$. This gives the category structure of S-$\mathfrak{ind}(\mathscr{A}, \mathscr{B})$.

For $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ indexed functors we define the horizontal composite $GF : \mathscr{A} \to \mathscr{C}$ as follows: For every $I$ in S, $(GF)^I = G^I F^I$ and for every $\alpha : J \to I$ the $\theta_\alpha$ for $GF$ is given in

$$\text{(Fc)}\qquad \begin{array}{ccccc} \alpha^*(GF)^I & & \xrightarrow{\quad\theta_\alpha\quad} & & (GF)^J\alpha^* \\ \| & & & & \| \\ \alpha^* G^I F^I & \xrightarrow[\theta_\alpha F^I]{} & G^J \alpha^* F^I & \xrightarrow[G^J \theta_\alpha]{} & G^J F^J \alpha^* \end{array}.$$

The identity indexed functor $1_{\mathscr{A}}$ is defined by $(1_{\mathscr{A}})^I = 1_{\mathbf{A}^I}$ with $\theta_\alpha = 1_{\alpha^*}$.

If $t : F \to F'$ and $u : G \to G'$, the horizontal composite $u * t : GF \to G'F'$ is defined in terms of the ordinary horizontal composition of natural transformations $(u * t)^I = u^I * t^I$.

The details showing that these operations do make S-$\mathfrak{ind}$ into a 2-category are straightforward and left to the reader.

To apply our coherence theorem for bicategories to S-indexed categories, we construct a bicategory $\mathfrak{B}$ as follows. The objects (0-cells) of $\mathfrak{B}$ are to be the objects of S *and* the objects of S-$\mathfrak{ind}$, and $\mathfrak{B}$ as a bicategory is to include the disjoint union of the 2-category S-$\mathfrak{ind}$ considered as a bicategory and the category S considered as a locally discrete bicategory (i.e. all 2-cells are identities). For each $A \in \mathbf{A}^I$ there is to be a morphism $A : I \to \mathscr{A}$, and for each $a : A \to A'$ in $\mathbf{A}^I$ a 2-cell $a : A \to A'$ in $\mathfrak{B}$. Thus $\mathfrak{B}(I, \mathscr{A}) = \mathbf{A}^I$. If $J \xrightarrow{\alpha} I \xrightarrow{A} \mathscr{A} \xrightarrow{F} \mathscr{B}$ are morphisms of $\mathfrak{B}$, we define $F \otimes A = F^I A$ and $A \otimes \alpha = \alpha^* A$. These extend canonically to functors $\otimes : \mathbf{A}^I \times \text{S-}\mathfrak{ind}(\mathscr{A}, \mathscr{B}) \to \mathbf{B}^I$ and $\otimes : [J, I] \times \mathbf{A}^I \to \mathbf{A}^J$. We also use $\otimes$ to denote composition in S and horizontal composition in S-$\mathfrak{ind}$.

We now show that $\mathfrak{B}$ is a bicategory.

In what follows we shall refer to the following diagram of morphisms of $\mathfrak{B}$

$$L \xrightarrow{\gamma} K \xrightarrow{\beta} J \xrightarrow{\alpha} I \xrightarrow{A} \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \xrightarrow{H} \mathscr{D}.$$

It is only necessary to define the maps $a$, $b$ and $c$ and check the conditions in those cases involving $A$, since S and S-$\mathfrak{ind}$ are already bicategories.

1. *Associativity (definition of a)*

$$\text{(i)}\qquad \begin{array}{ccc} (A \otimes \alpha) \otimes \beta & \xrightarrow{a_{A,\alpha,\beta}} & A \otimes (\alpha \otimes \beta) \\ \| & & \| \\ \beta^* \alpha^* A & \xrightarrow[\phi_{\alpha,\beta}(A)]{\cong} & (\alpha\beta)^* A \end{array},$$

(ii) $F\otimes$

$$\begin{array}{ccc}
(F\otimes A)\otimes\alpha & \xrightarrow{a_{F,A,\alpha}} & F\otimes(A\otimes\alpha) \\
\| & & \| \\
\alpha^{*}F^{I}A & \xrightarrow[\theta_\alpha(A)]{\cong} & F^{J}(\alpha^{*}A)\ ,
\end{array}$$

$$\begin{array}{ccc}
(G\otimes F)\otimes A & \xrightarrow{a_{G,F,A}} & G\otimes(F\otimes A) \\
\| & & \| \\
(GF)^{I}A & \xrightarrow{=} & G^{I}F^{I}A\ .
\end{array}$$

(iii) $G$

*laws* (*definition of b and c*)

$$\begin{array}{ccc}
A\otimes 1_I & \xrightarrow{b_A} & A \\
\| & & \| \\
1_I^{*}A & \xrightarrow[\psi_I(A)]{\cong} & A\ ,
\end{array}$$

$: 1_{\mathscr{A}}\otimes A = 1^{I}_{\mathscr{A}}A = A.$

*pentagon law* (B1)

(iv) $H$

$\otimes\alpha\otimes\beta\otimes\gamma$:

$$\begin{array}{ccc}
\gamma^{*}\beta^{*}\alpha^{*}A & \xrightarrow{\phi_{\beta,\gamma}\alpha^{*}} & (\beta\gamma)^{*}\alpha^{*}A \\
\downarrow{\scriptstyle \gamma^{*}\phi_{\alpha,\beta}} & & \\
\gamma^{*}(\alpha\beta)^{*}A & \boxed{\text{C1}} & \downarrow{\scriptstyle \phi_{\alpha,\beta\gamma}} \\
\downarrow{\scriptstyle \phi_{\alpha\beta,\gamma}} & & \\
(\alpha\beta\gamma)^{*}A & \xrightarrow{=} & (\alpha\beta\gamma)^{*}A\ ,
\end{array}$$

(ii) $F \otimes A \otimes \alpha \otimes \beta$:

$$\begin{array}{ccc}
\beta^*\alpha^*F^IA & \xrightarrow{\phi_{\alpha,\beta}F^I} & (\alpha\beta)^*F^IA \\
{\scriptstyle \beta^*\theta_\alpha}\downarrow & & \\
\beta^*F^J\alpha^*A & \boxed{\mathrm{F}} & \downarrow{\scriptstyle \theta_{\alpha\beta}} \\
{\scriptstyle \theta_\beta\alpha^*}\downarrow & & \\
F^K\beta^*\alpha^*A & \xrightarrow[F^K\phi_{\alpha,\beta}]{} & F^K(\alpha\beta)^*A \ ,
\end{array}$$

(iii) $G \otimes F \otimes A \otimes \alpha$:

$$\begin{array}{ccc}
\alpha^*(GF)^IA & \xrightarrow{\theta_\alpha} & (GF)^J\alpha^*A \\
{\scriptstyle =}\downarrow & & \\
\alpha^*G^IF^IA & \boxed{\mathrm{Fc}} & \downarrow{\scriptstyle =} \\
{\scriptstyle \theta_\alpha F^I}\downarrow & & \\
G^J\alpha^*F^IA & \xrightarrow[G^I\theta_\alpha]{} & G^IF^I\alpha^*A
\end{array}$$

(iv) $H \otimes G \otimes F \otimes A$:

$$\begin{array}{ccc}
((HG)F)^IA & \xrightarrow{=} & (HG)^IF^IA \\
{\scriptstyle =}\downarrow & & \\
(H(GF))^IA & & \downarrow{\scriptstyle =} \\
{\scriptstyle =}\downarrow & & \\
H^I(GF)^IA & \xrightarrow[=]{} & H^IG^IF^IA \ .
\end{array}$$

4. *Condition* (B2)

(i) $A \otimes 1_I \otimes \alpha$:

$$\begin{array}{ccc} \alpha^* 1_I^* A & \xrightarrow{\phi_{1_I,\alpha}} & (1_I\alpha)^* A \\ {\scriptstyle \alpha^*\psi_I} \searrow & \boxed{\text{C2}} & \swarrow {\scriptstyle =} \\ & \alpha^* A & \end{array},$$

(ii) $F \otimes 1_{\mathscr{A}} \otimes A$:

$$\begin{array}{ccc} (F1_{\mathscr{A}})^I A & \xrightarrow{=} & F^I 1^I_{\mathscr{A}} A \\ {\scriptstyle =} \searrow & & \swarrow {\scriptstyle =} \\ & F^I A & \end{array}.$$

Thus $\mathfrak{B}$ is a bicategory and all results about bicategories can be applied to it, thus yielding results about indexed categories. For example, for monoidal categories, the pentagon law and the law relating the left and right unit isomorphisms (our B1 and B2) imply two more conditions on the unit isomorphisms (see [10, p. 159]). Of course this extends trivially to bicategories giving the commutativity of

$$\begin{array}{ccc} (G \otimes F) \otimes 1_A & \xrightarrow{a} & G \otimes (F \otimes 1_A) \\ {\scriptstyle b} \searrow & & \swarrow {\scriptstyle G \otimes b} \\ & G \otimes F & \end{array}$$

and

$$\begin{array}{ccc} (1_C \otimes G) \otimes F & \xrightarrow{a} & 1 \otimes (G \otimes F) \\ {\scriptstyle c \otimes F} \searrow & & \swarrow {\scriptstyle c} \\ & G \otimes F & \end{array}.$$

If we apply the first of these conditions to the situations

$$I \xrightarrow{1_I} I \xrightarrow{A} \mathscr{A} \xrightarrow{F} \mathscr{B}$$

and

$$J \xrightarrow{1_J} J \xrightarrow{\alpha} I \xrightarrow{A} \mathscr{A}$$

in $\mathfrak{B}$ we get

$$\begin{array}{ccc} 1_I^* F^I A & \xrightarrow{\theta_{1,A}} & F^I 1_I^* A \\ {\scriptstyle \psi_I F^I A} \searrow & & \swarrow {\scriptstyle F^I \psi_I A} \\ & F^I A & \end{array}$$

and

$$\phi_{\alpha,1_J}(A)=\psi_J\alpha^*A : 1_J^*\alpha^*A\to\alpha^*A$$

which are F2 and C3 referred to above.

The canonical maps in $\mathfrak{B}$, when translated into indexed category notation, are those built out of $\phi, \psi, \theta$ and their inverses and $\alpha^*, F^I$, etc. For example both sides of the rectangle below are canonical maps (thus by the coherence theorem they are equal):

$$\begin{array}{ccc}
\beta^*G^J\alpha^*F^IA & \xrightarrow{\beta^*G^J\psi_J^{-1}\alpha^*F^IA} & \beta^*G^J1_J^*\alpha^*F^IA \\
\downarrow{\scriptstyle \beta^*\theta_\alpha^{-1}F^IA} & & \downarrow{\scriptstyle \beta^*\theta_{1_J}^{-1}\alpha^*F^IA} \\
\beta^*\alpha^*G^IF^IA & & \beta^*1_J^*G^J\alpha^*F^IA \\
\downarrow{\scriptstyle \phi_{\beta,\alpha}G^IF^IA} & & \downarrow{\scriptstyle \phi_{\beta,1_J}G^J\alpha^*F^IA} \\
(\alpha\beta)^*G^IF^IA & & \beta^*G^J\alpha^*F^IA \\
\downarrow{\scriptstyle \theta_{\alpha\beta}F^IA} & & \downarrow{\scriptstyle \theta_\beta\alpha^*F^IA} \\
G^K(\alpha\beta)^*F^IA & \xrightarrow[G^K\phi_{\beta,\alpha}^{-1}F^IA]{} & G^K\beta^*\alpha^*F^IA\ .
\end{array}$$

In the standard form of a word, all identities are dropped, the indexed functors come first each with its exponent, and then all the morphisms grouped together under the same star. For example, if none of $F, G, \alpha, \beta$ is an identity, then all vertices in the above diagram have the same standard form $G^KF^K(\alpha\beta)^*A$.

## 4. Coherence for indexed categories

The class of canonical maps obtained from the bicategory constructed in the previous section are not sufficient to deal with most of the situations arising in indexed categories. For example, when defining internal diagrams (see the next section), one of the conditions is $(\pi_1^*a)(\pi_0^*a)=\gamma^*a$. Strictly speaking this condition does not make sense because $\pi_1^*a$ and $\pi_0^*a$ do not compose, among other things. It is understood, however, that certain canonical maps must be introduced for it to make sense and, with a routine interpretation, there is only one way to do it. Thus the map between the codomain of $\pi_0^*a$ and the domain of $\pi_1^*a$ is

$$\gamma_0^*\partial_1^*A \xrightarrow{\text{can}} (\partial_1\pi_0)^*A=(\partial_0\pi_1)^*A \xrightarrow{\text{can}} \pi_1^*\partial_0^*A.$$

But for canonical maps to make sense we must be in $\mathfrak{W}$, i.e. the domains and codomains must be words, in this case all of length three. In $\mathfrak{W}$,

$$(\partial_1\pi_0)^*A = A\otimes(\partial_1\otimes\pi_0) \quad\text{and}\quad (\partial_0\pi_1)^*A = A\otimes(\partial_0\otimes\pi_1),$$

are two completely different words. Thus the above composite is not a canonical map. In fact the reason we constructed $\mathfrak{W}$ in the first place was exactly to avoid this sort of situation, where domains and codomains match up because of accidental equalities in $\mathfrak{B}$.

However, in the case of indexed categories, this equality was definitely intended and can hardly be called accidental. So we want to include this sort of equality, arising from equations in **S**, in our canonical maps for indexed categories.

For $\alpha: J\to I$ and $\beta: K\to J$ morphisms in **S**, the words $\alpha\otimes\beta$ and $\alpha\beta$ both evaluate to the same thing in $\mathfrak{B}$, namely $\alpha\beta$. Let $d_{\alpha,\beta}:\alpha\otimes\beta\to\alpha\beta$ be the unique 2-cell in $\mathfrak{W}$ which projects to the identity on $\alpha\beta$. All $d_{\alpha,\beta}$ are isomorphisms. Let $\mathfrak{D}$ be the smallest subbicategory of $\mathfrak{W}$ containing all objects and 1-cells of $\mathfrak{W}$ and all instances of $d_{\alpha,\beta}$ and $d_{\alpha,\beta}^{-1}$. Thus $\mathfrak{D}$ is the smallest class of 2-cells containing all instances of $a, b, c$ and $d$, their inverses, and closed under composition and $\otimes$.

A *canonical map*, in the context of indexed categories, is a 2-cell of $\mathfrak{D}$. From now on we use the term in this sense and if we want to refer to a canonical map arising from the bicategory structure only we will call it *bicanonical.*

**Theorem.** *To each morphism $W: A\to B$ in $\mathfrak{W}$ there is associated a morphism $\varrho W: A\to B$ and a canonical 2-cell $r_W: W\to\varrho W$. For $W, W': A\to B$ there is at most one canonical 2-cell $W\to W'$ and there is one if and only if $\varrho W=\varrho W'$.*

**Proof.** We call $\varrho W$ the *reduced standard form* of $W$ and it can be obtained from the standard form of $W$ by multiplying all the morphisms from **S** together (and then dropping any unnecessary identity which might arise). Thus, if $\sigma W= G\otimes(F\otimes(A\otimes(\alpha\otimes(\beta\otimes\gamma))))$, then $\varrho W=G\otimes(F\otimes(A\otimes\alpha\beta\gamma))$ if $\alpha\beta\gamma\neq 1$, and $\varrho W=G\otimes(F\otimes A)$ if $\alpha\beta\gamma=1$. If $W$ contains no (or only one) morphism from **S**, its standard form is already reduced. We define $r_W: W\to\varrho W$ by first finding a bicanonical map $W\to\bar{W}\otimes(\alpha_1\otimes(\alpha_2\otimes(\cdots\otimes\alpha_l)))$, where $\bar{W}$ is in standard form and contains no morphisms from **S**, and the $\alpha_i$ are morphisms from **S** none of which are identities (unless $W$ is of the form $\ulcorner 1_I\urcorner$ for $I\in\mathbf{S}$). $\bar{W}$ may be empty or $l$ may be 0, but not both. Now multiply all the $\alpha$'s together starting from the right. This gives $l-1$ maps containing a single $d$ each, whose composite we call $d$. It projects to an identity in $\mathfrak{B}$. Then, if $\alpha_1\alpha_2\cdots\alpha_n\neq 1$,

$$r_W = W\xrightarrow{\text{bican}}\bar{W}\otimes(\alpha_1\otimes(\alpha_2\otimes(\cdots\otimes\alpha_l)))\xrightarrow{\text{d}}\bar{W}\otimes(\alpha_1\alpha_2\cdots\alpha_l)$$

and, if $\alpha_1\alpha_2\cdots\alpha_n=1$,

$$r_W = W\xrightarrow{\text{bican}}\bar{W}\otimes(\alpha_1\otimes(\alpha_2\otimes(\cdots\otimes\alpha_l)))\xrightarrow{d}\bar{W}\otimes(\alpha_1\alpha_2\cdots\alpha_l)\xrightarrow{b_{\bar{W}}}\bar{W}.$$

Clearly the $r_W$ are canonical, so if $\varrho W = \varrho W'$, there is a canonical map $r_{W'}^{-1} r_W : W \to W'$. Conversely, by examining the domains and codomains of the elementary canonical maps (those containing only one instance of $a, b, c$ or $d$) we see that if there is a canonical map $W \to W'$, then $\varrho W = \varrho W'$.

We will show the uniqueness of the canonical map $W \to W'$ (under the assumption that $\varrho W = \varrho W'$) by showing that

$$(*) \qquad \begin{array}{ccc} W & \xrightarrow{\ \kappa\ } & W' \\ {\scriptstyle r_W}\downarrow & & \downarrow{\scriptstyle r_{W'}} \\ \varrho W & = & \varrho W' \end{array}$$

commutes for any canonical $\kappa$.

If $\kappa$ is bicanonical, then $\bar{W} = \bar{W}'$, $l = l'$, and $\alpha_i = \alpha_i'$, otherwise $W$ and $W'$ would not have the same standard form. Thus we get a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\qquad \kappa \qquad} & W' \\ {\scriptstyle \text{bican}}\downarrow & & \downarrow{\scriptstyle \text{bican}} \\ \bar{W} \otimes (\alpha_1 \otimes (\cdots \otimes \alpha_l)) & = & \bar{W} \otimes (\alpha_1 \otimes (\cdots \otimes \alpha_l)) \end{array}$$

which, if followed by $d$ (and $b_{\bar{W}'}$ if necessary), gives $r_{W'}\kappa = r_W$.

It is now sufficient to show that $(*)$ commutes for maps $\kappa$ involving a single instance of $d$ and no $a$, $b$, or $c$, since every canonical map is a composite of such maps, their inverses, and bicanonical ones. So assume that $\kappa$ is of this form, i.e. a tensor product of $d_{\beta,\gamma}$ with a number of identity 2-cells and with parentheses placed in some manner. A bicanonical map can be found to rearrange the parentheses in $\kappa$ and drop all identity 1-cells except those involving $\beta$ and $\gamma$, in such a way that $\kappa$ becomes

$$\begin{array}{l} V = \bar{W} \otimes (\alpha_1 \otimes (\alpha_2 \otimes (\cdots \otimes ((\beta \otimes \gamma) \otimes (\cdots \alpha_l))))) \\ {\scriptstyle \lambda}\downarrow \qquad\qquad \downarrow {\scriptstyle 1 \otimes (1 \otimes (1 \otimes (\cdots \otimes (d_{\beta,\gamma} \otimes (\cdots 1)))))} \\ V' = \bar{W} \otimes (\alpha_1 \otimes (\alpha_2 \otimes (\cdots \otimes (\beta\gamma \otimes (\cdots \alpha_l))))) \end{array}$$

where $\bar{W}$ is as before and no $\alpha_i$ is an identity. The bicanonical map is natural in its variables so

$$\begin{array}{ccc} W & \xrightarrow{\ \kappa\ } & W' \\ {\scriptstyle \text{bican}}\downarrow & & \downarrow{\scriptstyle \text{bican}} \\ V & \xrightarrow[\ \lambda\ ]{} & V' \end{array}$$

commutes, and since (∗) commutes for bicanonical maps it will commute for $\kappa$ if and only if it does for $\lambda$.

In calculating $r_V$ and $r_{V'}$, the bicanonical maps come from rearranging parentheses between maps from **S** and dropping identities also from **S**. The only situation where these do not project to identities in $\mathfrak{B}$ arises when there is only one map from **S** and it is an identity. In all other cases (∗) commutes because all maps involved project to identities in $\mathfrak{B}$, except possibly for a $b_{\bar{W}}$ tacked on at the end. The only other cases are when $\lambda$ is

$$\bar{W} \otimes d : \bar{W} \otimes (\beta \otimes \gamma) \to \bar{W} \otimes \beta\gamma$$

and $\beta = \gamma = \beta\gamma = 1$ or $\beta\gamma = 1$, $\beta, \gamma \neq 1$. It is easily seen that (∗) commutes in these cases too. □

We now introduce some notation which will be useful in our calculations with indexed categories. If $W$ and $V$ are morphisms $A \to B$ in $\mathfrak{WB}$ we write $W \bullet\!=\!\bullet V$ to mean that there is a canonical 2-cell $W \to V$ or equivalently $\varrho W = \varrho V$. If $f : W \to W'$ and $g : V \to V'$ we write $f \bullet\!=\!\bullet g$ if $W \bullet\!=\!\bullet V$ and $W' \bullet\!=\!\bullet V'$ and

$$\begin{array}{ccc} W & \xrightarrow{f} & W' \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ V & \xrightarrow[g]{} & V' \end{array}$$

commutes. We write $g \bullet f$ only if $W' \bullet\!=\!\bullet V$ and in that case

$$g \bullet f = \left( W \xrightarrow{f} W' \xrightarrow{\mathrm{can}} V \xrightarrow{g} V' \right).$$

Finally, if $U \bullet\!=\!\bullet W$ and $U' \bullet\!=\!\bullet W'$, we write $\bullet f \bullet : U \to U'$ and also $U \bullet \xrightarrow{f} \bullet U'$ to represent the 2-cell

$$U \xrightarrow{\mathrm{can}} W \xrightarrow{f} W' \xrightarrow{\mathrm{can}} U'.$$

The following proposition lists some of the main properties of these concepts. In these statements it is assumed that the domains and codomains match up so that the expressions make sense.

**Proposition.** *In the following statements* $f, f_1, f_2, g, g_1, g_2$ *represent 2-cells in* $\mathfrak{WB}$, $\alpha, \beta, \gamma$ *are morphisms in* **S**, *and F is an indexed functor.*

(1) $f_1 \bullet\!=\!\bullet f_2$ *and* $g_1 \bullet\!=\!\bullet g_2 \Rightarrow g_1 \bullet f_1 \bullet\!=\!\bullet g_2 \bullet f_2$,

(2) $f_1 \bullet\!=\!\bullet f_2 \Rightarrow F^I(f_1) \bullet\!=\!\bullet F^I(f_2)$,

(3) $g_1 \bullet\!=\!\bullet g_2 \Rightarrow \alpha^*(g_1) \bullet\!=\!\bullet \alpha^*(g_2)$,

(4) $F^I(g\cdot f)=F^I(g)\cdot F^I(f)$,

(5) $\alpha^*(g\cdot f)=\alpha^*(g)\cdot\alpha^*(f)$,

(6) $\alpha\beta=\gamma \Rightarrow \beta^*\alpha^*(f)\cdot{=}\cdot\gamma^*(f)$,

(7) $1_I^*(f)\cdot{=}\cdot f$,

(8) $\alpha^*F^I(f)\cdot{=}\cdot F^J\alpha^*(f)$,

(9) $\cdot f\cdot\cdot{=}\cdot f$. □

The relation $\cdot{=}\cdot$ may be read 'is canonically equivalent to'.

**Remark.** An alternate way of dealing with coherence for indexed categories, as Bénabou has pointed out, would be to pass to the corresponding fibration with a cleavage, where universal properties of cartesian morphisms would prove each diagram of canonical morphisms commutative. More on fibrations in this context can be found in [3].

## 5. The coherence theorem at work

Let

$$\mathbb{C}=\left(C_2 \;\begin{matrix}\xrightarrow{\pi_0}\\ \xrightarrow{\gamma}\\ \xrightarrow[\pi_1]{}\end{matrix}\; C_1 \;\begin{matrix}\xrightarrow{\partial_0}\\ \xleftarrow{\text{id}}\\ \xrightarrow[\partial_1]{}\end{matrix}\; C_0\right)$$

be a category object in **S**. Among other things this means that

(1) $\partial_0\pi_0=\partial_0\gamma$,

(2) $\partial_1\pi_1=\partial_1\gamma$,

(3) $\partial_0\pi_1=\partial_1\pi_0$,

(4) $\partial_0\text{id}=1_{C_0}=\partial_1\text{id}$.

**Definition.** If $\mathscr{A}$ is an indexed category, then an *internal diagram* of type $\mathbb{C}$ in $\mathscr{A}$ is a pair $(A,a)$ with $A$ an object of $\mathbf{A}^{C_0}$ and $a:\partial_0^*A\to\partial_1^*A$ in $\mathbf{A}^{C_1}$ such that

(1) $\text{id}^*a\cdot{=}\cdot 1_A$,

(2) $(\pi_1^*a)\cdot(\pi_0^*a)\cdot{=}\cdot\gamma^*a$.

This definition may seem somewhat mysterious at first but we are thinking of a functor $\mathbb{C}\to\mathscr{A}$, and there is a more or less mechanical means of translating definitions (and maybe even proofs) from ordinary category theory to indexed categories.

The idea is to express the definition (or proof) diagrammatically in terms of category objects in some sufficiently large category of sets (the Mitchell-Bénabou language is a machine for doing this), and then use cartesian closedness to translate these diagrams into a version which uses the basic elements of indexed categories (families, substitution, indexed functors, etc.). There may be several ways of doing this and only practice can tell which is the 'correct way'. There could be more than one correct translation, each expressing different aspects of the same concept. Now generalize to indexed categories and insert the unique canonical maps for which the formulas make sense.

Thus a functor $\mathbb{C}\to\mathbf{A}$ in *Set* consists of an object function and a morphism function which preserve domain, codomain, identities, and composition, i.e. we have functions $F_0 : C_0 \to \mathbf{A}_0$ and $F_1 : C_1 \to \mathbf{A}_1$ such that

$$
\begin{array}{ccccc}
C_2 & \overset{\pi_0,\ \gamma,\ \pi_1}{\rightrightarrows} & C_1 & \overset{\partial_0,\ \partial_1}{\underset{\mathrm{id}}{\rightleftarrows}} & C_0 \\
\Big\downarrow{\scriptstyle (F_1\pi_0, F_1\pi_1)} & & \Big\downarrow{\scriptstyle F_1} & & \Big\downarrow{\scriptstyle F_0} \\
\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 & \overset{\pi_0,\ \gamma,\ \pi_1}{\rightrightarrows} & \mathbf{A}_1 & \overset{\partial_0,\ \partial_1}{\underset{\mathrm{id}}{\rightleftarrows}} & \mathbf{A}_0
\end{array}
$$

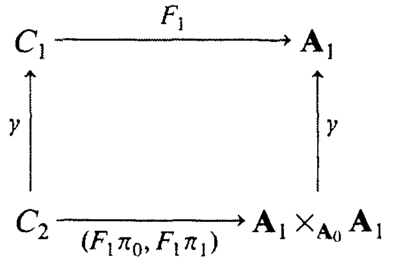
commutes in the well-known sense. We can't generalize a function $F_0 : C_0 \to \mathbf{A}_0$ to indexed categories, so we transpose and get $1 \to \mathbf{A}_0^{C_0}$, i.e. an object $A$ of $\mathbf{A}^{C_0}$. Transposing $F_1 : C_1 \to \mathbf{A}_1$ gives us $1 \to \mathbf{A}_1^{C_1}$ i.e. a morphism $a$ of $\mathbf{A}^{C_1}$. The commutative diagrams $(i = 0, 1)$

$$
\begin{array}{ccc}
C_0 & \xrightarrow{F_0} & \mathbf{A}_0 \\
\Big\uparrow{\scriptstyle \partial_i} & & \Big\uparrow{\scriptstyle \partial_i} \\
C_1 & \xrightarrow[F_1]{} & \mathbf{A}_1
\end{array}
$$

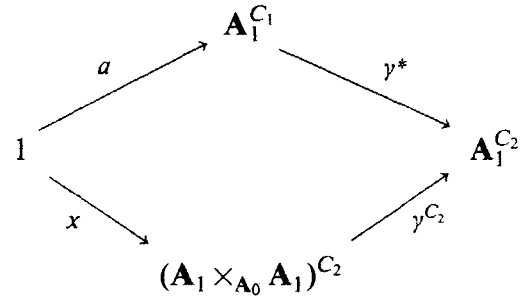
transpose to give

$$
\begin{array}{ccccc}
 & & \mathbf{A}^{C_0} & & \\
 & \overset{A}{\nearrow} & & \overset{\partial_i^*}{\searrow} & \\
1 & & & & \mathbf{A}_0^{C_1} \\
 & \underset{a}{\searrow} & & \underset{\partial_i^{C_1}}{\nearrow} & \\
 & & \mathbf{A}_1^{C_1} & &
\end{array}
$$

i.e. $\mathrm{dom}(a)=\partial_0^*A$ and $\mathrm{cod}(a)=\partial_1^*A$. Thus $a:\partial_0^*A\to\partial_1^*A$. Similarly, the preservation of identities gives $\mathrm{id}^*(a)=1_A$. The condition




becomes




where $x$ followed by the isomorphism $(\mathbf{A}_1\times_{\mathbf{A}_0}\mathbf{A}_1)^{C_2}\xrightarrow{\cong}\mathbf{A}_1^{C_2}\times_{\mathbf{A}_0^{C_2}}\mathbf{A}_1^{C_2}$ is given by

$$1\xrightarrow{a}\mathbf{A}_1^{C_1}\xrightarrow{(\pi_0^*,\pi_1^*)}\mathbf{A}_1^{C_2}\times_{\mathbf{A}_0^{C_2}}\mathbf{A}_1^{C_2}.$$

The inverse of this isomorphism followed by $\gamma^{C_2}$ gives composition in $\mathbf{A}^{C_2}$, so we get the condition $(\pi_1^*a)(\pi_0^*a)=\gamma^*a$. Inserting the canonical maps gives us the definition of internal diagram.

**Proposition 1.** *Indexed functors $F:\mathscr{A}\to\mathscr{B}$ carry internal diagrams of type $\mathbb{C}$ in $\mathscr{A}$ to internal diagrams of type $\mathbb{C}$ in $\mathscr{B}$.*

**Proof.** Let $(A,a)$ be an internal diagram in $\mathscr{A}$. Consider $(F^{C_0}A,\bullet F^{C_1}a\bullet)$ where $\bullet F^{C_1}a\bullet:\partial_0^*F^{C_0}A\to\partial_1^*F^{C_0}A$. Then

$$\mathrm{id}^*(\bullet F^{C_1}a\bullet)\bullet=\bullet\,\mathrm{id}^*(F^{C_1}a)\bullet=\bullet F^{C_0}\mathrm{id}^*(a)\bullet=\bullet F^{C_0}1_A=1_{F^{C_0}A}.$$

Also

$$\begin{aligned}\pi_1^*(\bullet F^{C_1}a\bullet)\bullet\pi_0^*(\bullet F^{C_1}a\bullet)\bullet&=\bullet\,\pi_1^*(F^{C_1}a)\bullet\pi_0^*(F^{C_1}a)\\ \bullet=\bullet F^{C_0}\pi_1^*(a)\bullet F^{C_0}\pi_0^*(a)\bullet&=\bullet F^{C_0}(\pi_1^*(a)\bullet\pi_0^*(a))\\ \bullet=\bullet F^{C_0}\gamma^*(a)\bullet&=\bullet\,\gamma^*(F^{C_1}a)\bullet=\bullet\,\gamma^*(\bullet F^{C_1}a\bullet).\end{aligned}$$

Thus $(F^{C_0}A,\bullet F^{C_1}a\bullet)$ is an internal diagram of type $\mathbb{C}$ in $\mathscr{B}$. □

This calculation may be read as a set of instructions for inserting canonical morphisms; the result is a diagram with 15 morphisms which commutes in virtue of the commutativity of 10 smaller diagrams.

Our 'machine' can also be used to translate the definition of natural transformation $t: F \to F'$ into the following definition.

**Definition.** If $(A, a)$ and $(A', a')$ are internal diagrams of type $\mathbb{C}$ in $A$, then a *morphism* $f: (A, a) \to (A', a')$ is a morphism $f: A \to A'$ in $\mathbf{A}^{C_0}$ such that

$$\begin{array}{ccc} \partial_0^* A & \xrightarrow{\partial_0^* f} & \partial_0^* A' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a'} \\ \partial_1^* A & \xrightarrow[\partial_1^* f]{} & \partial_1^* A' \end{array}$$

commutes.

**Proposition 2.** *If $f: (A, a) \to (A', a')$ is a morphism of internal diagrams and $F$ is an indexed functor $\mathscr{A} \to \mathscr{B}$, then $F^{C_0} f$ is a morphism of internal diagrams $(F^{C_0}A, \bullet F^{C_1}a \bullet) \to (F^{C_0}A', \bullet F^{C_1}a' \bullet)$.*

**Proof.** Apply $F^{C_1}$ to the diagram of the definition to get

$$\begin{array}{ccc} F^{C_1}\partial_0^* A & \xrightarrow{F^{C_1}\partial_0^* f} & F^{C_1}\partial_0^* A' \\ {\scriptstyle F^{C_1}a}\downarrow & & \downarrow{\scriptstyle F^{C_1}a'} \\ F^{C_1}\partial_1^* A & \xrightarrow[F^{C_1}\partial_1^* f]{} & F^{C_1}\partial_1^* A' \end{array}$$

which gives

$$\begin{array}{ccc} \partial_0^* F^{C_0} A & \xrightarrow{\partial_0^* F^{C_0} f} & \partial_0^* F^{C_0} A' \\ \bullet & & \bullet \\ {\scriptstyle F^{C_1}a}\downarrow & & \downarrow{\scriptstyle F^{C_1}a'} \\ \bullet & & \bullet \\ \partial_1^* F^{C_0} A & \xrightarrow[\partial_1^* F^{C_0} f]{} & \partial_1^* F^{C_0} A' \end{array}$$

which is what we wanted. □

An *internal functor* $\phi: \mathbb{D} \to \mathbb{C}$ between category objects in $\mathbf{S}$ consists of three morphisms $\phi_i: D_i \to C_i$ $(i = 0, 1, 2)$ such that

$$\begin{array}{ccccc} D_2 & \Rrightarrow & D_1 & \rightleftarrows & D_0 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ C_2 & \Rrightarrow & C_1 & \rightleftarrows & C_0 \end{array}$$

commutes in the usual sense.

**Proposition 3.** *If $(A,a)$ is an internal diagram of type $\mathbb{C}$ in $\mathscr{A}$ and $\phi:\mathbb{D}\to\mathbb{C}$ is an internal functor between category objects, then $(\phi_0^*A, \bullet\phi_1^*a\bullet)$ is an internal diagram of type $\mathbb{D}$ in $\mathscr{A}$, where $\bullet\phi_1^*a\bullet : \partial_0^*\phi_0^*A\to\partial_1^*\phi_0^*A$.*

**Proof.**

$$\mathrm{id}^*(\bullet\phi_1^*a\bullet)\bullet = \bullet\,\mathrm{id}^*\phi_1^*a\bullet = \bullet\,\phi_0^*\mathrm{id}^*a\bullet = \bullet\,\phi_0^*1_A = 1_{\phi_0^*A}.$$

$$\pi_1^*(\bullet\phi_1^*a\bullet)\bullet\pi_0^*(\bullet\phi_1^*a\bullet)\bullet = \bullet(\pi_1^*\phi_1^*a)\bullet(\pi_0^*\phi_1^*a)$$

$$\bullet = \bullet(\phi_2^*\pi_1^*a)\bullet(\phi_2^*\pi_0^*a)\bullet = \bullet\,\phi_2^*(\pi_1^*a\bullet\pi_0^*a)$$

$$\bullet = \bullet\,\phi_2^*\gamma^*a\bullet = \bullet\,\gamma^*\phi_1^*a\bullet = \bullet\,\gamma^*(\bullet\phi_1^*a\bullet). \qquad \square$$

We now give a more complicated example where it is necessary to keep track of the words involved. We want to define the indexed functor category $\mathscr{A}^{\mathscr{B}}$ and show that the evaluation functor $E:\mathscr{B}\times\mathscr{A}^{\mathscr{B}}\to\mathscr{A}$ is indexed. For this we need some preliminary results. We omit most of the proofs which are easy verifications.

If $X$ is an object of $\mathbf{S}$ and $\mathscr{A}$ an indexed category, then $\mathbf{A}^X$ can be made into an indexed category. The definition is easily guessed by generalizing the situation for sets. Let $(\mathbf{A}^X)^I = \mathbf{A}^{X\times I}$ and for $\alpha: J\to I$ let $\alpha^*:(\mathbf{A}^X)^I\to(\mathbf{A}^X)^J$ be $(X\times\alpha)^*:\mathbf{A}^{X\times I}\to\mathbf{A}^{X\times J}$. The $\phi_{\alpha,\beta}$ and $\psi_I$ for $\mathbf{A}^X$ are taken to be the $\phi_{X\times\alpha,X\times\beta}$ and $\psi_{X\times I}$ for $\mathbf{A}$ respectively.

If $\xi: Y\to X$, we can make $\xi^*:\mathbf{A}^X\to\mathbf{A}^Y$ into an indexed functor by defining $(\xi^*)^I$ to be $(\xi\times I)^*$. The $\theta_\alpha$ for $\xi^*$ is given at $A\in\mathbf{A}^{X\times I}$ by the canonical morphism

$$\theta_\alpha(A) = \mathrm{can}: (Y\times\alpha)^*(\xi\times I)^*\to(\xi\times J)^*(X\times\alpha)^*.$$

Condition (F), when evaluated at $A$, becomes commutativity of a diagram containing only canonical maps between words whose reduced standard form is $(\xi\times\alpha\beta)^*A$. It commutes by virtue of the coherence theorem.

If $Z\xrightarrow{\eta}Y\xrightarrow{\xi}X$ are morphisms of $\mathbf{S}$, then $\phi_{\xi,\eta}:\eta^*\xi^*\to(\xi\eta)^*$ extends to an indexed isomorphism, given at $I$ by $\phi_{\xi\times I,\eta\times I}$.

We now define the *indexed functor category* $\mathscr{A}^{\mathscr{B}}$. What should $(\mathscr{A}^{\mathscr{B}})^I$ be? For category objects in **Set** an object of $(\mathbf{A}^{\mathbf{B}})^I$ can be viewed in any of the equivalent ways

$$\frac{\dfrac{\dfrac{1\to(\mathbf{A}^{\mathbf{B}})^I}{I\to\mathbf{A}^{\mathbf{B}}}}{\mathbf{B}\times I\to\mathbf{A}}}{\mathbf{B}\to\mathbf{A}^I}\ .$$

The last line generalizes to indexed categories and so we use this as our definition. Thus $(\mathbf{A}^{\mathscr{B}})^I$ is the category whose objects are indexed functors $\mathscr{B}\to\mathscr{A}^I$ and whose morphisms are indexed natural transformations between them. For $\alpha: J\to I$, $\alpha^*:(\mathbf{A}^{\mathscr{B}})^I\to(\mathbf{A}^{\mathscr{B}})^J$ is given by composition with the indexed functor $\alpha^*:\mathscr{A}^I\to\mathscr{A}^J$. Thus, if $F:\mathscr{B}\to\mathscr{A}^I$ is an object of $(\mathbf{A}^{\mathscr{B}})^I$, then

$$\alpha^*(F)=\alpha^*F=\left(\mathscr{B}\xrightarrow{F}\mathscr{A}^I\xrightarrow{\alpha^*}\mathscr{A}^J\right),$$

and if $t:F\to F'$ is a morphism of $(\mathbf{A}^{\mathscr{B}})^I$, then $\alpha^*(t)=\alpha^*t$. Since S-ind is a 2-category, $\alpha^*$ is indeed a functor. We define $\phi_{\alpha,\beta}:\beta^*\alpha^*\to(\alpha\beta)^*$ at $F$ by

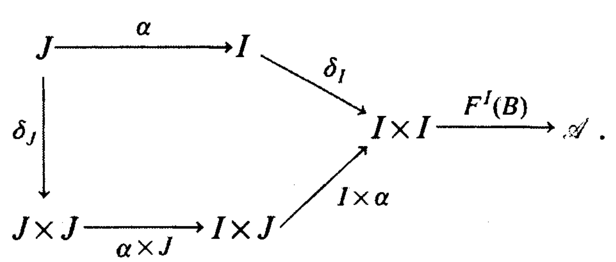
$$\phi_{\alpha,\beta}(F)=\phi_{\alpha,\beta}F:\beta^*\alpha^*F\to(\alpha\beta)^*F.$$

Similarly $\psi_I(F)=\psi_IF$. Checking (IC1) and (IC2) is immediate once we evaluate at $F$ and raise to the $I$.

Define a functor $E^I:\mathbf{B}^I\times(\mathbf{A}^{\mathscr{B}})^I\to\mathbf{A}^I$ as follows: For $B\in\mathscr{B}^I$ and $F:\mathscr{B}\to\mathscr{A}^I$, let $E^I(B,F)=\delta_I^*F^I(B)$, where $\delta_I:I\to I\times I$ is the diagonal morphism. $E^I$ is easily seen to be a functor if we set $E^I(b,t)=\delta_I^*(t^I(B')F^I(b))$ for $b:B\to B'$ and $t:F\to F'$. For $\alpha:J\to I$ we define $\theta_\alpha(B,F):\alpha^*E^I(B,F)\to E^J\alpha^*(B,F)$ to be the morphism

$$\alpha^*\delta_I^*F^I(B)\xrightarrow{\text{can}}\delta_J^*(\alpha\times J)^*(I\times\alpha)^*F^I(B)\xrightarrow{\delta_J^*(\alpha\times J)^*\theta_\alpha(B)}\delta_J^*(\alpha\times J)^*F^J(\alpha^*B).$$

Here the canonical map is between the words $(F^I(B)\otimes\delta_I)\otimes\alpha$ and $((F^I(B)\otimes(I\times\alpha))\otimes(\alpha\times J))\otimes\delta_J$, whose letters are pictured in the diagram




Note that, *although $\delta_J^*(\alpha\times J)^*\theta_\alpha(B)$ appears to be canonical, it is not*. We may consider $(\alpha\times J)^*\theta_\alpha(B)$ to be $(\alpha^*)^J\theta_\alpha(B)$ and as such it is canonical between two associations of

$$J\xrightarrow{\alpha}I\xrightarrow{B}\mathscr{B}\xrightarrow{F}\mathscr{A}^I\xrightarrow{\alpha^*}\mathscr{A}^J$$

but there is no way to attach $\delta_J:J\to J\times J$ and make the whole thing canonical. Thus $\delta_J^*$ is merely a functor applied to a canonical map.

**Proposition 4.** *The $E^I$ and $\theta_\alpha$ defined above are the data for an indexed functor $E:\mathscr{B}\times\mathscr{A}^{\mathscr{B}}\to\mathscr{A}$.*

**Proof.** We wish to check condition (F) for $E$, i.e. we want to show the commutativity of

$$\begin{array}{ccc}
\beta^*\alpha^*E^I(B,F) & \xrightarrow{\phi_{\alpha,\beta}(E^I(B,F))} & (\alpha\beta)^*E^I(B,F) \\
\Big\downarrow{\scriptstyle \beta^*\theta_\alpha(B,F)} & & \\
\beta^*E^J(\alpha^*B,\alpha^*F) & & \Big\downarrow{\scriptstyle \theta_{\alpha\beta}(B,F)} \\
\Big\downarrow{\scriptstyle \theta_\beta(\alpha^*B,\alpha^*F)} & & \\
E^K(\beta^*\alpha^*B,\beta^*\alpha^*F) & \xrightarrow[E^K(\phi_{\alpha,\beta}(B),\phi_{\alpha,\beta}F)]{} & E^K((\alpha\beta)^*B,(\alpha\beta)^*F)\,.
\end{array}$$

So as to avoid confusion about which maps are canonical we must view the domains and codomains as words. Thus we introduce the following notation which allows us to read the words literally.

Let

$$A_1=F^I(B):I\times I\to\mathscr{A},$$
$$A_2=F^J(\alpha^*B):I\times J\to\mathscr{A},$$
$$A_3=F^K(\beta^*\alpha^*B):I\times K\to\mathscr{A},$$
$$A_4=F^K((\alpha\beta)^*B):I\times K\to\mathscr{A},$$
$$a_1=\theta_\alpha(B):(I\times\alpha)^*A_1\to A_2,$$
$$a_2=\theta_\beta(\alpha^*B):(I\times\beta)^*A_2\to A_3,$$
$$a_3=\theta_{\alpha\beta}(B):(I\times\alpha\beta)^*A_1\to A_4,$$
$$a_4=F^K(\phi_{\alpha,\beta}(B)):A_3\to A_4.$$

With this notation the expression for $\theta_\alpha(B,F)$ is

$$\alpha^*\delta_I^*A_1\xrightarrow{\text{can}}\delta_J^*(\alpha\times J)^*(I\times\alpha)^*A_1\xrightarrow{\delta_J^*(\alpha\times J)^*a_1}\delta_J^*(\alpha\times J)^*A_2,$$

i.e.

$$\theta_\alpha(B,F)\bullet=\bullet\,\delta_J^*(\alpha\times J)^*a_1.$$

Similarly,

$$\theta_\beta(\alpha^*B,\alpha^*F)\bullet=\bullet\,\delta_K^*(\beta\times K)^*(\alpha\times K)^*a_2,$$
$$\theta_{\alpha\beta}(B,F)\bullet=\bullet\,\delta_K^*(\alpha\beta\times K)^*a_3$$

and

$$E^K(\phi_{\alpha,\beta}(B),\phi_{\alpha,\beta}F)\bullet=\bullet\,\delta_K^*(\alpha\beta\times K)^*a_4.$$

Thus we want to show that

$$(\delta_K^*(\alpha\beta\times K)^*a_4)\bullet(\delta_K^*(\beta\times K)^*(\alpha\times K)^*a_2)\bullet(\beta^*\delta_J^*(\alpha\times J)^*a_1)\bullet=\bullet\,\delta_K^*(\alpha\beta\times K)^*a_3$$

(we can leave off the $\phi_{\alpha,\beta}(\delta_I^*A_1)$ on the right as it is canonical).

Condition (F) for the functor $F$ gives

$$a_4 a_2 (I\times\beta)^* a_1 \bullet{=}\bullet a_3$$

to which we can apply $\delta_K^*(\alpha\beta\times K)^*$. The result now follows from the resulting relation by making the substitutions indicated by

$$\delta_K^*(\alpha\beta\times K)^* a_2 \bullet{=}\bullet \delta_K^*(\beta\times K)^*(\alpha\times K)^* a_2$$

and

$$\delta_K^*(\alpha\beta\times K)^*(I\times\beta)^* a_1 \bullet{=}\bullet \beta^*\delta_J^*(\alpha\times J)^* a_1.$$

Thus $E:\mathscr{B}\times\mathscr{A}^{\mathscr{B}}\to\mathscr{A}$ is indexed. $\square$

Similar calculations show that $E$ establishes an equivalence of categories

$$\mathbf{S}\text{-}\mathfrak{ind}(\mathscr{C},\mathscr{A}^{\mathscr{B}})\to\mathbf{S}\text{-}\mathfrak{ind}(\mathscr{B}\times\mathscr{C},\mathscr{A})$$

$$\Phi\mapsto E(1_B\times\Phi).$$

In this sense, $E$ is the indexed evaluation functor.

As Bénabou has pointed out, this result was incorrectly stated in [13, p. 61]. Indeed, the isomorphisms in the statements of (1.1.1), (1.1.2), (1.1.3), and (1.1.4) should have been equivalences of categories.

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